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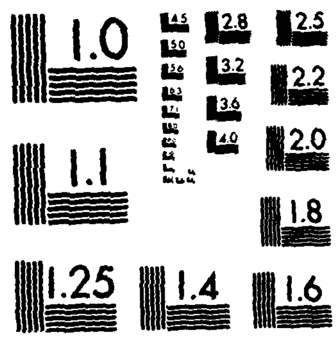

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ESTIMATING IFRA AND NBU SURVIVAL CURVES

BASED ON CENSORED DATA

by

Jane-Ling Wang

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Estimating IFRA and NBU Survival Curves

Based on Censored Data

Running Title: Estimating IFRA and NBU Survival Curves

Jane-Ling Wang*
Division of Statistics
University of California, Davis

ABSTRACT

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<u>Key words and phrases</u>: Star-shaped function, superadditive function product-limit estimator, hazard function.

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1. INTRODUCTION AND SUMMARY

In reliability theory and survival analysis, it is often desirable to estimate survival curves or equivalently, life distributions. In some circumstances the lifetime X_i of the i^{th} item is not observed, rather we only know that it exceeded a time Y_i. For example, in clinical trials patients may move away or die of some other causes and therefore are lost to the study. Let (X_i, Y_i), $i=1, \ldots, n$ be i.i.d. random variables with X_i and Y_i independent for each i. Let $F(t) = P(X_1 < t)$ and $G(t) = P(Y_1 < t)$ denote the distribution function of X_1 and Y_1 respectively. F(t) is referred to as the life distribution and G, the censoring distribution. In random censorship model one observes (Z_i, δ_i), $i=1, \ldots, n$, where $Z_i = \text{Min}\ (X_i, Y_i)$, $\delta_i = I(X_i < Y_i)$ and, I(A) is the indicator function of a set A. Techniques for estimating F(t) using (Z_i, δ_i), $i=1, \ldots, n$, have been known for a long time only recently has there been much concern with estimating F(t) when it is known to belong to a certain nonparametric class of distributions.

A variety of such classes which arise naturally in practice are given in Barlow and Proschan (1981). Lo and Phadia (1984) treat the classes of convex distributions and increasing failure rate distributions. We shall consider two nonparametric classes in this paper: (1) the class of distributions with increasing failure rate average (IFRA) and (2) the class of distributions with the "New Better than Used (NBU)" property.

For a life distribution F(t), let $\bar{F}(t) = 1-F(t)$ and $H(t) = -\log \bar{F}(t)$ denote its survival and hazard function respectively. A distribution function F(t) with $F(0) = 0$ is said to be IFRA if H(t) is starshaped, that is; $H(t)/t$ is a nondecreasing function of t. A distribution function F(t) is said to be NBU if $\bar{F}(x+y) < \bar{F}(x)\ \bar{F}(y)$, or equivalently $H(x+Y) > H(x) + H(Y)$; that is, H(t) is superadditive. Note that an IFRA distribution is also NBU.

The class of IFRA distributions arises naturally in shock models (Esary, Marshall and Proschan (1973)) and is the smallest class containing the exponential distributions, closed under the formation of coherent structure and taking limits in distributions (Birbaum, Esary and Marshall (1966)). The NBU concepts means that the residual lifetime of a used item tends to be shorter than the lifetime of a new item.

The product-limit estimator (PL-estimator) $F_n(t)$ due to Kaplan and Meier (1958), the most commonly used nonparametric estimator of F, will now be defined. Let $Z_{(i)}$ be the i^{th} order statistics from the sample $\{Z_i, i=1,\ldots,n\}$ and $\delta_{(i)}$ be the corresponding indicator function associated with $Z_{(i)}$. The PL-estimator is defined by

$$(1.1) \qquad 1 - F_n(t) = \begin{cases} \prod\limits_{Z_{(i)} \leq t} \left(\frac{n-i}{n-i+1}\right)^{\delta_{(i)}} & , \text{ if } t < Z_{(n)} \\ 0 & , \text{ otherwise.} \end{cases}$$

Large sample properties of the PL-estimator have been studied extensively by Breslow and Crowley (1974), Csörgő and Horwáth (1983), Gill (1983) and Lo and Singh (1984) among others. Moreover, Wellner (1982) showed that the PL-estimator posseses desirable asymptotic optimality properties, e.g. minimax. In this paper we modify the PL-estimator (1.1) so that is has the known property (IFRA or NBU) of F(t) and remains close to the original PL-estimator.

We shall restrict the estimation problem only on compact intervals [0,T], where T is any point with $F(T) < 1$ and $G(T) < 1$. Let $H_n(t) = -\log(1-F_n(t))$ be the hazard function of $F_n(t)$. In Section 2, we construct estimators $C_n(D_n)$ of the hazard function $H(t)$ under the assumption that F is IFRA (NBU) by modifying $H_n(t)$. The modified estimator of F(t) is the distribution function whose hazard function is $C_n(t)$ ($D_n(t)$) respectively. The modified estimators C_n and D_n can be expressed in close form (cf. (2.1) and (2.3)) and are easy to compute. Moreover,

$$\sup_{0<t<T} | C_n(t) - H(t) | < \sup_{0<t<T} | H_n(t) - H(t) | .$$

Hence $C_n(t)$ is closer to $H(t)$ than $H_n(t)$ in the sense of Kolmogorov distance.

The main results of the paper are Theorems 4.1 and 4.2 where we show that under mild conditions,

$\sup_{0<t<T} n^{1/2} | C_n(t) - H_n(t) |$ and $\sup_{0<t<T} n^{1/2} | D_n(t) - H_n(t) |$ tend to zero in probability. This implies then the asymptotic behavior of our modified estimators are the same as that of the PL-estimator.

The proofs of Theorems 4.1 and 4.2 utilize an i.i.d. representation of $H_n(t)$ by Lo and Singh (1984). Relevant results are summarized in Section 3.

2. DESCRIPTION OF THE ESTIMATORS

In this section we shall modify the PL-estimator (defined in (1.1)) so it has the desired IFRA or NBU property. We shall construct the estimators on the interval $[0,T]$.

First consider the case when F is known to be IFRA.

Let $C_n(t) = \sup\{h(t) : h(t) < H_n(t)$ for $0 < t < T$, where $h(t)$ is starshaped on $[0,T]\}$, i.e. $C_n(t)$ is the greatest starshaped minorant (GSM) of $H_n(t)$ on $[0,T]$. It is easy to check that $C_n(t)$ is starshaped. A closed form of C_n can be obtained as follows:

Let $m = \sum_{i=1}^{n} I(\delta_i = 1, Z_i < T)$ be the number of uncensored observations in the interval $[0,T]$ and $\{Z_1^0, Z_2^0, \ldots, Z_m^0\}$ be the uncensored observations from $\{Z_1, \ldots, Z_n\}$ which correspond to actual lifetimes, and are in the interval $[0,T]$. The PL-estimator assigns positive mass on the interval $[0,T]$ only to those points $\{Z_i^0, i=1, \ldots, m\}$. Let α_j be the smallest slope of all the lines connecting the origin and the point $(Z_k^0, H_n(Z_k^0-))$, for $k=j, j+1, \ldots, m+1$, where,

$Z_{m+1}^0 = T$, H_n is taken to be right continuous and hence $H_n(Z_k^0-) = H_n(Z_{k-1}^0)$.

That is, $\alpha_j = \min\limits_{j \leq k \leq m+1} \{H_n(Z_k^0-)/Z_k^0\}$.

Using the results of Wang (1984a) in the i.i.d. case, it is easy to check that the modified estimator C_n, which is the GSM of H_n is a piecewise linear function of the form:

$$C_n(t) = \alpha_j t \ , \text{ for } Z_{j-1}^0 \leq t < Z_j^0 \ , \ j=1, \ldots, m+1. \tag{2.1}$$

Thus $C_n(t)$ has a close form expression and is easy to compute. The following lemmas asserts that C_n is closer to any starshaped function on $[0,T]$ than H_n, and hence is closer to the true hazard function $H(t)$ than H_n.

Lemma 2.1. For any starshaped function $\phi(t)$ on $[0,T]$,

$$\sup_{0 \leq t \leq T} | C_n(t) - \phi(t) | \leq \sup_{0 \leq t \leq T} | H_n(t) - \phi(t) | .$$

PROOF: Let $\sup\limits_{0 \leq t \leq T} | H_n(t) - \phi(t) | = A.$

If $A = \infty$, there is nothing to be proved. If $A < \infty$, $\phi(t) - A$ is starshaped on $[0,T]$ and $H_n(t) \geq \phi(t) - A$. From the definition of C_n we have $C_n(t) \geq \phi(t) - A$.

Since C_n is a minorant of H_n, we have $C_n(t) < H_n(t) < \phi(t) + A$, so the lemma holds. ■

Since H is starshaped we have

<u>Corollary 2.1</u> $\sup_{0<t<T} | C_n(t) - H(t) | < \sup_{0<t<T} | H_n(t) - H(t) |$.

Thus C_n is a better estimator of H than the hazard function, H_n, of the PL-estimator. A modified estimator of $F(t)$ is then taken to be the distribution $\tilde{F}_n(t) = 1 - \exp\{-C_n(t)\}$. Note that $\tilde{F}_n$ is not necessarily a better estimate of $F(t)$ than F_n although its hazard function is a better estimate of $H(t)$ than H_n. Theorem 4.1 however guarantees its equivalence to the PL-estimator.

Next, we shall consider the case when $F(t)$ is assumed to be NBU. Since an NBU distribution has superadditive hazard function. It is therefore natural to consider

$$(2.2) \qquad D_n(t) = \inf_{0<t<T} \{H_n(s+t) - H_n(s)\} \quad ,$$

as our estimator of the hazard function $H(t)$. The estimator D_n is analogous to the estimator in Boyles and Samaniego (1984) developed for the uncensoring case. Their results show that $D_n(t)$ is superadditive. Since H_n is a step function with jumps at $\{Z_1^0, Z_2^0, \ldots, Z_m^0\}$, to compute D_n one only needs to take the infinum in (2.2) over these points, i.e.,

$$(2.3) \qquad D_n(t) = \inf_{0<i<m} \{H_n(Z_i^0 + t) - H_n(Z_i^0)\}, \text{ where } Z_0^0 = 0,$$

and D_n is a step function with jumps at points of the form $Z_r^0 - Z_s^0$ for some r and s. Note that $D_n(t) < H_n(t)$ for all t, and if H_n is superadditive, $D_n(t) = H_n(t)$ for all t. To estimate $F(t)$, we again use the distribution

$F_n^*(t)$ whose hazard function is D_n, i.e., $F_n^*(t) = 1 - \exp\{-D_n(t)\}$.

Unlike the IFRA case we cannot show that D_n is a better estimator of H than H_n. However we have

Lemma 2.2. For any superadditive function $\phi(t)$ on [0,T],

$$\sup_{0<t<T} \left| D_n(t) - \phi(t) \right| < 2 \sup_{0<t<T} \left| H_n(t) - \phi(t) \right|.$$

PROOF: Let $\sup_{0<t<T} \left| H_n(t) - \phi(t) \right| = A.$

It then follows that

$$\begin{aligned} D_n(t) &= \inf_{0<s<T} \{H_n(s+t) - H_n(s)\} \\ &> \inf_{0<s<T} \{\phi(s+t) - \phi(s) - 2A\} \\ &= \phi(t) - 2A \quad , \end{aligned}$$

where the last step follows from the superadditivity of $\phi(t)$. The lemma now follows from the fact that

$$D_n(t) < H_n(t) < \phi(t) + A. \qquad \blacksquare$$

Lemma 2.2 implies that D_n is a strongly uniformly consistent estimator of H(t) on [0,T] with at least the same rate of convergence as $H_n(t)$. Theorems 4, 5 and 6 of Boyles and Samaniego (1984) are immediate consequence of Lemma 2.2.

3. PRELIMINARIES

As mentioned in Section 1, the techniques of our main results (Theorems 4.1 and 4.2) are based on the i.i.d. representations of the PL-process derived by Lo and Singh (1984). In this section, we shall give the results that are later needed in Section 4 to establish the main theorems.

Lemma 3.1. (Lo and Singh (1984)) Let $\eta_1, \ldots, \eta_n$ be i.i.d. random variables with mean zero, variance σ^2 and $|\eta_i| < C$ for some constant C for all $1<i<n$. For any positive d and z satisfying $Cz < d$ and $nz\sigma^2 < d^2$ one has

$$P(\ |\sum_{i=1}^{n} \eta_i\ | < 3d) < 2\ e^{-z}\ .$$

PROOF: This is Lemma 1 of Lo and Singh (1984). The proof is based upon Markov's inequality and Taylor's expansions. ∎

Let $Q(t)$ and $Q_1(t)$ be the distribution (subdistribution) function such that $\bar{Q}(t) = \bar{F}(t)\ \bar{G}(t)$ and $Q_1(t) = P(Z_1 < t \text{ and } \delta_1 = 1)$. It is easy to check that $Q_1(t) = \int_0^t \bar{G}(t)\ d\ F(t)$, and hence $dQ_1(t) = \bar{G}(t)dF(t)$. Note that $Q(t)$ is the distribution of the observation Z. Let $g(t) = \int_0^t [\bar{Q}(s)]^{-2}\ dQ_1(s)$ be the asymptotic variance of H_n (Breslow and Crowley (1976)). For positive real z and t, and δ taking values 0 or 1, let $\xi\ (z,\delta,t) = [\bar{Q}(z)]^{-1}\ I\ (z<t \text{ and } \delta=1) - g(z\Lambda t)$. The following lemma, although not stated explicitely in Lo and Singh (1984), can be proved by tracing the arguments of their Theorem 1. The proof can be found in Proposition 1 of Lo, Mack and Wang (1985).

Lemma 3.2. If F is continuous, for any $P > 0$, there exists constant $\theta > 0$ (depending on P) such that

(3.1) $H_n(t) - H(t) = n^{-1} \sum_{i=1}^{n} \xi\ (Z_i,\ \delta_i,\ t) + R_n(t)$, where

(3.2) $P(\ \sup_{0<t<T}\ |\ R_n(t)\ | > \theta(\text{lon}\ n/n)^{3/4}\ ) = O(n^{-P})$.

We shall now assume that F(t) has bounded density function f(t) on [0,T], and let $M = \text{Sup}\{|f(t)| : 0<t<T\}$. Let

$$\xi_i(t) = \xi(Z_i, \delta_i, t), \text{ and } q = \bar{Q}(T) > 0.$$

<u>Lemma 3.3</u>. (a) For any x in $[0,T]$ and $\varepsilon < (3/2)Mx$,

$$(3.3) \quad P\{|\, n^{-1} \sum_{i=1}^{n} \xi_i(x)\,| > \varepsilon\} < 2 \exp\{-\frac{n\varepsilon^2 q^2}{9Mx}\}\ .$$

(b) For any $x < y$ both in $[0,T]$ and $\varepsilon < (3/4)M(y-x)$,

$$(3.4) \quad P\{|n^{-1} \sum_{i=1}^{n} [\xi_i(x) - \xi_i(y)]\,| > \varepsilon\} < 2 \exp\{-\frac{n\varepsilon^2 q^2}{9M(y-x)}\}\ .$$

<u>PROOF:</u> In order to facilitate the aplication of Lemma 3.1, let $\eta_i = \xi_i(x)$. Then $E\,\eta_i = 0$ and

$$\sigma^2 = \text{Var}(\eta_i) = g(x)$$

$$= \int_0^x [\bar{Q}(t)]^{-2}\, d\, Q_1(t)$$

$$= \int_0^x [\bar{Q}(t)]^{-2}\, \bar{G}(t)\, dF(t), \text{ since } dQ_1(t) = \bar{G}(t)\, dF(t),$$

$$< F(x) q^{-2}$$

$$< Mxq^{-2}\ .$$

It can be checked easily that

$|\xi_i(x)| < [\bar{Q}(x)]^{-1} + [\bar{Q}(x)]^{-2} < 2q^{-2}$. Taking $C = 2q^{-2}$, $d = (n\varepsilon)/3$ and $z = (n\varepsilon^2 q^2)/(9Mx)$, then $Cz < d$, and $nz\sigma^2 = d^2$, so Lemma 3.1 applies to $|\sum_{i=1}^{n} \eta_i|$ yielding (3.3).

To show (3.4) let $\eta_i = \xi_i(x) - \xi_i(y)$. Again $E\eta_i = 0$ and

$$\begin{aligned}
\mathrm{Var}\ (\eta_i) &= g(x) + g(y) - 2g(x\wedge y) \\
&= \int_x^y [\bar{Q}(t)]^{-2}\, dQ_1(t) \\
&= \int_x^y [\bar{Q}(t)]^{-2}\, \bar{G}(t)\, dF(t) \\
&< [F(y) - F(x)]\, q^{-2} \\
&< M(y-x)\, q^{-2}.
\end{aligned}$$

When $C = 4q^{-2}$, $d = (n\varepsilon)/3$ and $z = (n\varepsilon^2 q^2)/[9M(y-x)]$, the conditions of Lemma 3.1 are satisfied and (3.4) follows. ■

4. MAIN RESULTS

In this section we shall show the $n^{1/2}$-equivalence of the PL-estimator F_n and the modified estimator $\tilde{F}_n$ and F_n^*. We shall assume that $F(t)$ has bounded density $f(t)$ on $[0,T]$, and $M = \sup_{0<t<T} |f(t)|$. Let $a_F = \mathrm{Inf}\ \{x\colon F(x) > 0\}$ be the left endpoint of the support of F, and $L = T - a_F$.

The proofs of the main results utilize notions of linear interpolating functions which we now define.

Let $\{k_n\}$ be a sequence of integers tending to infinity. For each n, partition the interval $[a_F, T]$ into k_n equal length subintervals $[a_j^n, a_{j+1}^n]$, $j=0, 1, \ldots, k_n-1$, where $a_0^n = a_F$, $a_{k_n}^n = T$.

For any function g on $[a_F,T]$ define its linear interpolating function $L_n g$ as

$L_n g(a_j^n) = g(a_j^n)$ for $j=0, 1, \ldots, k_n$, and $L_n g(x)$ is linear on $[a_j^n, a_{j+1}^n]$ for each $j=0, 1, \ldots, k_n-1$.

Thus $L_n g$ is a piecewise linear function. Note that $\{L_n H_n(t) : 0<t<T\}$ is a stochastic process whose realizations are piecewise linear functions.

Let A_n be the event that L_nH_n is starshaped on $[0,T]$ and B_n be the event that L_nH_n is superadditive on $[0,T]$.

We proceed with a series of Lemmas.

<u>Lemma 4.1</u>. If there exists a constant $\lambda > 0$ such that for any $0 < x < y < T$,

$$[H(y)/y] - [H(x)/x] > \lambda(y-x), \tag{4.1}$$

then for any $p > 0$, and $k_n = 0((n/\log n)^{3/8})$,

$$1-P(A_n) < 4k_n \exp\{-\frac{n\, q^2\lambda^2L^3}{36Mk_n^3}\} + 0(n^{-p}) \text{ , for n sufficiently large.}$$

<u>PROOF</u>: Since L_nH_n is piecewise linear, it is starshaped on $[0,T]$ if and only if $H_n(a_j^n)/a_j^n$ is increasing in j. This implies that $1 - P(A_n) < \sum_{j=1}^{k_n} P(E_j)$, where E_j is the event that $H_n(a_j^n)/a_j^n$ is greater than $H_n(a_{j+1}^n)/a_{j+1}^n$.

To compute $P(E_j)$ for fixed $j>1$, let $x = a_j^n$, $y= a_{j+1}^n$ and hence $y-x = L/k_n$ and $y>x>(L/k_n)$. Now consider

$$\begin{aligned}
P(E_j) &= P\{[H_n(y)/y] < [H_n(x)/x]\} \\
&= P\{[H_n(y) - H(y)] - [H_n(x) - H(x)] < H(x) - H(y) + H_n(x)\ (y-x)/x\} \\
&< P\{[H_n(y) - H(y)] - [H_n(x) - H(x)] < [H_n(x) - H(x)]\ (y-x)/x - \lambda y(y-x)\}, \\
&\quad \text{from (4.1)}, \\
&< P\{[H_n(y) - H(y)] - [H_n(x) - H(x)] < -(\lambda/2)y(y-x)\} \\
&\quad + P\{[H_n(x) - H(x)]\ (y-x)/x > (\lambda/2)y(y-x)\} \\
&< P\{|\frac{1}{n}\sum_{i=1}^{n}[\xi_i(y) - \xi_i(x)]| > (\lambda/4)y(y-x)\} + \\
&\quad P\{|R_n(x) - R_n(y)| > (\lambda/4)y(y-x)\} + P\{|\frac{1}{n}\sum_{i=1}^{n}\xi_i(x)| > (\lambda/4)xy\}
\end{aligned}$$

$$+ P\{|R_n(x)| > (\lambda/4)xy\}$$

$$< P\{|\frac{1}{n}\sum_{i=1}^{n}[\xi_i(x) - \xi_i(y)]| > (\lambda/2)\ (y-x)^2\} + P\{|R_n(x)\ | > \ (\lambda/4)(L/k_n)^2\}$$

$$+ P\{|R_n(y)\ | > (\lambda/4)(L/k_n)^2\} + P\{\ |\frac{1}{n}\sum_{i=1}^{n}\xi_i(x)\ | \ > (\lambda/2)\ x(y-x)\}$$

$$+ P\{|R_n(x)| > (\lambda/2)\ (L/k_n)^2\}, \text{ since } y > a_2^n = 2(y-x) = 2(L/k_n).$$

For $k_n = 0\ ((n/\log n)^{3/8})$, Lemma 3.2 implies that

$$\text{(4.2)} \qquad P\{\sup_{0<t<T} |\ R_n(t)\ | > (\lambda/4)\ (L/k_n)^2\} = 0(n^{-P}).$$

Since $y-x = L/k_n$ tends to zero as n tends to ∞, we have $(\lambda/2)\ x\ (y-x) <$ $(3/2)Mx$ and $(\lambda/2)\ (y-x)^2 < (3/4)\ M\ (y-x)$ for n sufficiently large.

Lemma 3.3 now implies that

$$\text{(4.3)} \quad P\{|\frac{1}{n}\sum_{i=1}^{n}\xi_i(x)| > (\lambda/2)\ x\ (y-x)\} < 2\exp\{\frac{nq^2\lambda^2x(y-x)^2}{36M}\}$$

$$< 2\exp\{-\frac{nq^2\lambda^2L^3}{36Mk_n^3}\}\ \text{, and}$$

$$\text{(4.4)} \quad P\{|\frac{1}{n}\sum_{i-1}^{n}[\xi_i(x) - \xi_i(y)]| > (\lambda/2)\ (y-x)^2\} < 2\exp\{-\frac{nq^2\lambda^2(y-x)^3}{36M}\}$$

$$< 2\exp\{-\frac{nq^2\lambda^2L^3}{36Mk_n^3}\} \quad .$$

The lemma now follows from (4.2), (4.3) and (4.4), where we replaced $k_n0(n^{-P})$ by $0(n^{-P})$ since P can be any positive number. ■

Next we obtain a similar result for the event B_n. We first find a necessary and sufficient condition for the polygon $L_n H_n(t)$ to be superadditive using the results of Bruckner (1960, 1962). Note that {0} $\{a_j^n,\ j=0,\ 1,\ \ldots,\ k_n\}$ are the vertices of the polygon $L_n H_n(t)$.

Lemma 4.2. $L_n H_n(t)$ is superadditive on $[0,T]$ if and only if it is superadditive on $\{a_j^n,\ j=0,\ 1,\ \ldots,\ k_n\}$.

PROOF: We only need to consider the case when $L_n H_n(t)$ is superadditive on $\{a_j^n,\ j=0,\ 1,\ \ldots,\ k_n\}$.

Let $f_1(t) = 0$, for $0 < t < a_F$, and

$f_2(t) = L_n H_n(t + a_F)$, for $0 < t < L$.

Then $f_2(t)$ is a polygon on $[0,L]$ with equally spaced vertices, and Theorem 8 of Bruckner (1960) implies that $f_2(t)$ is superadditive on $[0,L]$. Following the definitions of Bruckner (1962, P. 127), it can be checked easily that f_1 and f_2 form a decomposition of $L_n H_n(t)$. Since the minimal superadditive extension of f_1 is the zero function on $[0,T]$, Theorem 1 of Bruckner (1962) implies that $L_n H_n(t)$ is superadditive on $[0,T]$. ∎

Lemma 4.3. If there exists a constant $\alpha > 0$ such that for any $0<x<y<T$,

(4.5) $\quad H(x+y) > H(x) + H(y) + \alpha xy$, then for any $P > 0$ and

$k_n = 0\ ((n/\log n)^{3/8})$,

$$1 - P\ (B_n) < 4\ k_n^2 \exp\{-\ \frac{nq^2\alpha^2L^3}{72Mk_n^3}\} + O(n^{-P}) \quad \text{for } n \text{ sufficiently large.}$$

PROOF: Lemma 4.2 implies that,

$$1 - P(B_n) = P\{ \text{There exists } i \text{ and } j \text{ such that}$$

$$H_n(a_i^n) + H_n(a_j^n) > H_n(a_i^n + a_j^n) \}$$

$$< \sum_{i,j>1}^{k_n} P(E_{ij}) \text{ , where } E_{ij} \text{ is the event}$$

$$\text{that } H_n(a_i^n) + H_n(a_j^n) > H_n(a_i^n + a_j^n) \text{ .}$$

To evaluate $P(E_{ij})$ for fixed i and j, let $a_i^n = x$, $a_j^n = y$.

$$P(E_{ij}) = P\{H_n(x) - H(x) + H_n(y) - H(y) > H_n(x+y) - [H(x) + H(y)]\}$$

$$< P\{H_n(x) - H(x) + H_n(y) - H(y) > H_n(x+y) - H(x+y) + \alpha xy\}$$

$$= P\{[H_n(x) - H(x)] - [H_n(x+y) - H(x+Y)] > - [H_n(y) - H(y)] + \alpha xy\}$$

$$< P\{[H_n(x) - H(x)] - [H_n(x+y) - H(x+y)] > (\alpha/2)xy\}$$

$$+ P\{H_n(y) - H(y) > (\alpha/2)xy\}$$

$$< P\{\left| \frac{1}{n} \sum_{i=1}^{n} [\xi_i(x) - \xi_i(x+y)]\right| > (\alpha/4)(L/k_n)y\} + P\{\left| \frac{1}{n} \Sigma\, \xi_i(y)\right|$$

$$> (\alpha/4)(L/k_n)y\} + 3\, P\{ \sup_{0<t<T} |R_n(t)| > (\alpha/4)(L/k_n)^2\}$$

$$< 2 \exp \{ - \frac{nq^2\alpha^2(L/k_n)^2y)}{144M} \} + 2 \exp \{ - \frac{nq^2\alpha^2(L/k_n)^2y}{144M} \} + O(n^{-p}) \text{ ,}$$

by Lemma 3.3 since $(\alpha/4)(L/k_n)y < (3/2)\, My < (3/4)\, My$ for large n,

$$< 4 \exp \{ - \frac{nq^2\alpha^2L^3}{72Mk_n^3} \} + O(n^{-p}) \quad \text{, since } y>2(L/k_n).$$

Lemma 4.3 thus follows. Here we use the same fact that $k_n^2\, O(n^{-p})$ can be replaced by $O(n^{-p})$ since the choice of p can be arbitrary. ■

The following two Lemmas are propositions 2 and 3 of Wang (1982).

Lemma 4.4. For any distribution function F with hazard function H,

$$\sup_{0<t<T} n^{1/2} \mid H(t) - L_nH(t) \mid \to 0 \text{ implies that}$$

$$\sup_{0<t<T} n^{1/2} \mid H_n(t) - L_nH_n(t) \mid \to 0 \text{ in probability.}$$

Lemma 4.5. Let H be a differentiable hazard function satisfying

(4.6) $|H'(x) - H'(y)| < \beta \; |x-y|$, for any x, y in [0,T], and some constant $\beta>0$, then

$$\sup_{0<t<T} |H(t) - L_nH(t)| < 2\beta \; (L/k_n)^2.$$

We are now ready to prove the main theorems.

Theorem 4.1

Let F be an IFRA distribution function satisfying (4.1) and (4.6), then

$$\sup_{0<t<T} n^{1/2} \; |C_n(t) - H_n(t)| \to 0 \text{ in probability.}$$

PROOF: Let $k_n = [\frac{nq^2\lambda^2L^3}{108M \log n}]^{1/3}$.

Setting P=3 in Lemma 4.1 implies that $1-P(A_n) < n^{-2}$ for n sufficiently large. When A_n occurs L_nH_n is starshaped so Lemma 2.1 implies that

$$\sup_{0<t<T} |C_n(t) - H_n(t)| < \sup_{0<t<T} \mid C_n(t) - L_nH_n(t) \mid + \sup_{0<t<T} \mid L_nH_n(t) - H_n(t) \mid$$

$$< 2 \sup_{0<t<T} \mid L_nH_n(t) - H_n(t) \mid.$$

Since $n^{1/4} \; k_n^{-1} \to 0$, Lemma 4.5 implies that

$$\sup_{0<t<T} n^{1/2} \mid H(t) - L_nH(t) \mid \to 0.$$

The theorem now follows from Lemma 4.4. ■

Corallary 4.1. Under the assumptions of Theorem 4.1,

$$\sup_{0<t<T} n^{1/2} \mid \tilde{F}_n(t) - F_n(t) \mid \to 0 \text{ in probability.}$$

PROOF: $$\sup_{0<t<T} \mid \tilde{F}_n(t) - F_n(t) \mid = \mid \exp(-C_n(t)) - \exp(-H_n(t)) \mid$$
$$< \mid C_n(t) - H_n(t) \mid.$$ ■

Theorem 4.2. Let F be an NBU distribution function satisfying (4.5) and (4.6), then

$$\sup_{0<t<T} n^{1/2} \mid D_n(t) - H_n(t) \mid \to 0 \text{ in probability.}$$

PROOF: The proof follows the derivation of Theorem 4.1 utilizing Lemmas 2.2, 4.3, 4.4 and 4.5, and choosing $k_n = [\frac{nq^2\alpha^2L^3}{216M \log n}]^{1/3}$. ■

Corallary 4.2. Under the assumption of Theorem 4.2,

$$\sup_{0<t<T} n^{1/2} \mid F_n^*(t) - F_n(t) \mid \to 0 \text{ in probability.}$$

PROOF: The proof is similar to Corallary 4.1. ■

Remarks:

1. Condition (4.1) essentially means that H is uniformly strictly starshaped, or the life distribution F is uniformly strictly IFRA. Condition (4.5) is a uniformly strict superadditivity condition on the hazard function H, which essentially means that the life distribution F should be uniformly strictly NBU.

2. From the proof of Theorems 4.1 and 4.2, it can be seen that $\sup_{0<t<T} n^{1/2} \mid L_nH_n(t) - H_n(t) \mid$ also tends to zero in probability for

properly chosen $\{k_n\}$ (e.g. the choices of k_n in Theorems 4.1. and 4.2.). A practical consequence is that, instead of computing $C_n(D_n)$ exactly, one may be able to use L_nH_n as our estimator of the hazard function. The estimtor L_nH_n is much easier to compute than $C_n(D_n)$ and has a high probability ($P(A_n)$ or $P(B_n)$) of being starshaped (NBU).

3. The techniques of this paper can also be applied to obtain analogues of Theorems 4.1 and 4.2 for distributions with Decreasing Failure Rate Average (DFRA) or distributions which are New Worse than Used (NWU). Similar results to Theorems 4.1 and 4.2 can be obtained for DFRA and NBU distributions respectively.

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Jane-Ling Wang
Division of Statistics
University of California
Davis, CA 95616

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